The Quantum Hall Effect and Relativistic Hydrogen Atom

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Abstract
In two-dimensional electron gas when a large magnetic field is applied in one direction and an electric field perpendicular to it, there is a current in a direction perpendicular to both. This current is called the Hall effect. It remained without quantization until 1980 when it was found that the quantization leads to the correct measurement of $\hbar/e^2$. Therefore, the quantized Hall effect was further studied at high magnetic fields where fractional quantization is found. The fractional charge can arise from the “incompressibility” in the flux quantization. Laughlin wrote a wave function, the excitations of which are fractionally charged quasiparticles. This wave function comes in competition with charge density waves but for a few fractions it does not give the ground state. If incompressibility is not considered, and it is allowed to be compressible, the fractional charge can arise from the angular momentum which appears in the Bohr magneton in the form of g values. Usually the positive spin is considered but we consider the positive as well as the negative values so that there is spin-charge coupling. The values thus calculated for the fractional charge agree with the experimental data on the quantum Hall effect. The fractional charge requires both the positive as well as negative sign for the spin. When this idea is applied to the hydrogen atom, it is necessary to perform the calculation using the Dirac equation, so that in a relativistically correct treatment, the Bohr’s $-1/n^2$ type eigenvalues are seriously affected. Multiple values for the charge of the electron emerge and the eigenvalues of the hydrogen atom then depend on the effective electron charge.

Introduction

The Hall effect was discovered in 1878. It shows that the resistivity of a metal or a semiconductor is a linear function of the magnetic field. In 1980 it was found by von Klitzing[1] that when the samples are two dimensional, at low temperatures, there is a plateau in the linear plot which gives the correct value of the $\hbar/e^2$, the ratio of the Planck’s constant and the square of the electron charge. Later on, Laughlin obtained the wave function on the basis of first principles[2,3] and used the “flux quantization”. The experiments were repeated by using smaller samples at very low temperatures and much higher fields by Tsui, Stormer and Gossard [4] who found plateaus in the Hall current at fractional values of the charge, particularly, $(1/3)e$ was found. Later experimental work[5] also found the value of $2/3$. Several years of hard work resulted in the discovery of more than 148 fractions. It was reported by Shrivastava[6-9] that angular momentum theory can be used to derive the fractional charges found in the experimental data. Shrivastava’s calculated values are the same as those present in the Stormer’s
experimental measurements given in the Nobel prize lecture[5]. In the U. S. many authors made a variety of efforts to predict or to model the fractional values of the electron charge.

In the present paper, we show that the fractional charges can be predicted by the use of Dirac equation. We also examine the relativistic solution of the eigen values of the hydrogen atom and show the effect of the fractional charge. We find the relativistic values of the eigen values which depend on the electron charge.

The Hall effect.

When electric field is applied along the x direction and the magnetic field along the z direction, there is a current along the y direction. This is called the Hall current. If the sample is made into a sheet, such that it is thin along the y direction, a large Hall voltage can be measured. The resistivity along the xy plane is a linear function of magnetic field,

$$\rho_{xy} = \frac{B}{n_{sc}}$$  \hspace{1cm} (1)

where $n = n_s/A$ is the number of electrons per unit area. A plot of $\rho_{xy}$ as a function of the magnetic field, B, therefore, measures the electron density. According to flux quantization, $B.A = \phi_0$ where $\phi_0 = hc/e$ is the unit flux. Substituting the quantized flux in the expression for the resistivity, we can show that,

$$\rho_{xy} = \frac{h}{n_s e^2}.$$ \hspace{1cm} (2)

The cyclotron frequency is $\omega = eB/mc$ so that,

$$\frac{h}{2\pi mc} = eV_y.$$ \hspace{1cm} (3)

Multiplying both sides by $e/h$,

$$\frac{e^2 B}{2\pi mc} = \frac{e^2}{h} V_y.$$ \hspace{1cm} (4)

Introducing $(1/2)g$ on both sides,

$$\frac{1}{2} g \frac{e^2}{2\pi mc} = \frac{1}{2} g \frac{e^2}{h} V_y.$$ \hspace{1cm} (5)

the resistivity becomes,

$$\rho_{xy} = \frac{h}{1/2 ge^2}.$$ \hspace{1cm} (6)

Here the g value has come from $g\mu_B B.S$ where $S_z = 1/2$ so that when the electron charge in the Bohr magneton, $\mu_B = e\hbar/2mc$ is redefined as $e_{\text{eff}} = \frac{1}{2} ge$, the effective charge appears instead of the electron charge in the resistivity expression (6). The introduction of the “flux quantization” in the old Hall effect formula immediately yields the “quantized Hall effect”. We are thus ready with the integer as well as the fractional quantized Hall effect.
in a single theory. So what is new? We have introduced both signs for the spin, positive spin as well as negative spin in writing,

\[
\frac{1}{2} g_\pm = \frac{l + \frac{1}{2} \pm s}{2l + 1}
\]

the derivation of which is given in the book[8]. For \( s=+1/2 \), we get,

\[
\nu_+ = \frac{1}{2} g_+ = \frac{l + 1}{2l + 1}
\]

and for \( s=-1/2 \) (negative spin) we get,

\[
\nu_- = \frac{1}{2} g_- = \frac{l}{2l + 1}
\]

A tabulation of the \( \nu_\pm \) for various values of \( l \) gives the fractional charges correctly. For example, eq.(9) gives \( 1/3 \) for \( l=1 \), so that the electron charge becomes \( (1/3)e \).

**The Dirac equation.**

The Dirac equation is based on the symmetry of space and time variables and the constancy of the velocity of light. The kinetic energy is therefore, linear in the momentum as given below,

\[
(c\alpha.p + \beta mc^2)\psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t)
\]

where \( m \) is the rest mass of the electron, \( p \) is the linear momentum, \( c \) is the velocity of light and \( \alpha_x, \alpha_y, \alpha_z \) and \( \beta \) are the anticommuting matrices. The free particle solutions are,

\[
E_\pm(p) = \pm (c^2p^2 + m^2c^4)^{1/2}
\]

where the positive sign is associated with the electron and the negative sign with the positron. A classical electron moving in a circular orbit has an orbital angular momentum, \( L=rp \) and a magnetic moment, \( \mu=-erv/2 \). Therefore, the ratio of the magnetic moment to the dipole moment is \( \gamma=\mu/L=-e/2m \). Since the electron has spin, it produces a correction to the ratio of the magnetic moment to the dipole moment so that the corrected value becomes \( \gamma=-ge/2m \). Actually, the Dirac equation does not conserve angular momentum unless spin is introduced. Therefore, the spin effects are assigned to the Dirac equation. The spin angular momentum is \( \hbar/2 \). The electron is a charged particle so that the electromagnetic field is associated with it. The correction to the \( g \) value due to this effect is given in terms of the fine structure constant as \( g=2(1+\alpha/2\pi) \) where \( \alpha=e^2/4\pi\hbar c=1/137.03599976(50) \). The electromagnetic correction to the \( g \) value is written as \( g-2=\alpha/\pi=0.00231930043737(82) \). All of the measurements of the \( g \) value, done to great accuracy are limited to the electron in \( L=0 \) state only. In the finite \( L \) states, such attention has not been paid. Even then, it is clear that \( g \) is related to the charge of the electron. We have seen below eq.(9) that the fractional charge arises from the \( g \) values. 

We introduce the electromagnetic field so that \( p \) changes to \( p-(e/c)A \) and \( e\phi \) is to be added to the potential. Thus the introduction of the electromagnetic field is equivalent to
inclusion of A and \( \varphi \), the vector and the scalar potentials. Therefore the Dirac Hamiltonian becomes,
\[
\mathcal{H} = \beta mc^2 + c \alpha . (p - \frac{e}{c} A) + e \varphi (x, t). \tag{12}
\]

The wave function is four dimensional but it is sufficient to take only two at a time so that writing only the electron part and excluding positrons we get,
\[
\left[ \frac{1}{2m} \sum |p - \frac{e}{c} A(x, t)|^2 - \frac{\varepsilon h}{2mc} \sum \sigma_j B_j(x) \right] \Psi = (E - mc^2) \Psi. \tag{13}
\]

Therefore the magnetic moment of the particle is \( \varepsilon h/2mc \). The magnetic moment of the proton calculated using mass of the proton does not agree with the experimental value unless the g value of the proton is included in the formula. Accordingly, the proton magnetic moment is correctly obtained by using the expression,
\[
\mu_{\text{proton}} = \frac{g_p \varepsilon h}{2mc}, \text{ with } g_p = 5.58 \text{ for the proton.}
\]

It may be argued that the charge of the proton has become \( g_p e \) with positive value. Similarly, the magnetic moment of the neutron is correctly determined by introducing the g value for the neutron, \( g_n = -3.826 \) but the charge of the neutron is explained by its decay into electron, proton and the antineutrino. The theory of eq.(9) which determines the \((1/3)e\) charge of the electron is thus consistent with that used for the proton and the neutron.

**The Relativistic Hydrogen atom.**

The relativistic hydrogen atom was solved by Sommerfeld[10] in 1916. In recent years Martinez-y-Romero[11,12,13], Katsura and Aoki[14], and Lombardi[15] have discussed the solutions theoretically while the interest on accurate experimental measurements of Lamb shift has also continued[16,17]. We show that the effective charge of the electron through the g values gives corrections to the eigen values of the hydrogen atom. According to the Sommerfeld’s solution for the potential, \( V(r) = -Z e^2/r \), \( \gamma = Ze^2/hc \), the eigen values are given by,
\[
E = mc^2 \left[ 1 + \frac{\gamma^2}{(s+n'^2)^2} \right]^{1/2} \tag{14}
\]

Expanding the square root, the solution becomes,
\[
E = mc^2 \left[ 1 - \frac{\gamma^2}{2n'^2} - \frac{\gamma^4}{2n^4} \left( \frac{n}{|k|} - \frac{3}{4} \right) \right] \tag{15}
\]

\( n = n' + |k|, \ k = j(j+1) - l(l+1) + \frac{1}{4} \) which gives \( k = l+1 \) for \( j = l + \frac{1}{2} \) and \( k = -l \) for \( j = l - \frac{1}{2} \).

Thus there are two values of \(|k|\), both give two eigen values. The third term in the eigen value is,
\[
E^{(3)} = -mc^2 \frac{\gamma^4}{2n^3 |k|} \tag{16}
\]

This is the term which gives \(-1/n^3\) type correction to the Bohr’s formula which had only \(-1/n^2\) term. Here \( n = n' + k = n' + [j(j+1) - l(l+1) + \frac{1}{4}] \), so that there are two values of \( n \).
(i), \( n=n'(l+1) \) for \( j=l+1/2 \)  \hspace{1cm} \text{(17a)}

(ii) \( n=n'-l \) for \( j=l-1/2 \). \hspace{1cm} \text{(17b)}

For positive sign, \( n=n'+(j-\frac{1}{2})+1 \) \hspace{1cm} \text{(18a)}

and for negative sign \( n=n'-(j+\frac{1}{2}) \). \hspace{1cm} \text{(18b)}

In our theory \( g(2l+1)=2j+1 \). Therefore, \( j=\frac{1}{2} g(2l+1) \), the substitution of which in eq.(18) makes the eigen values depend on \( g \) and hence on the charge, because \( \frac{1}{2} g e \) is the charge of the electron.

**Conclusions**

We are able to derive the fractional charges from the theory of angular momentum. The calculated values are in accord with the experimental measurements of quantum Hall effect. Our method of deriving fractional charges also gives the correct magnetic moments of the proton as well as that of the neutron. The relativistic corrections to the eigen values of the hydrogen atom are found to depend on the effective charge of the electron.

**References**