i.e. We have losses instead of amplification. Incident beam $I_0$ is being attenuated.

Alternatively,

We can write them interim of

$$I( x + dx ) = I(x) + \gamma I(x) \, dx$$
$$= I(x) + dI$$

(difference equation)

Solving it,

$$dI = \gamma I \, dx$$

$$\frac{dI}{dx} = \gamma I$$

$$I(x) = I(0) e^{\gamma x}$$

$$\gamma = \sigma N = (cm^2) \frac{bil}{cm^3} = bil/cm$$

But if $N_2 > N_1$ $\Rightarrow \frac{dI}{dt}$ or $\frac{dI}{dx}$ are positive and the light is amplified.

This condition ($N_2 > N_1$) is not sufficient must include losses.

Note: for amplification to occur, the stimulated emission rate of energy increase must be greater than the rate of energy losses.

Losses $\rightarrow$ specified by cavity lifetime or cavity loss time or photon lifetime.

**Cavity life time ($t_c$)**

$T_c$ is the time taken for the intensity of light introduced into the cavity to decay to $1/e$ of its value.
At $t=0$, energy density $\rho(\nu)$ traveling backwards and forwards between the mirrors (absorption and leakage occurs) energy density $\rho(t)$ at $t$,

$$\rho(t) = \rho(0)e^{-t/\tau_c}$$

Or,

$$\frac{d\rho}{dt} = -\frac{1}{\tau_c}\rho(0)e^{-t/\tau_c} = -\frac{\rho(t)}{\tau_c}$$

i.e.,

$$\frac{d\rho(t)}{\rho(t)} = \frac{dt}{\tau_c} = \alpha$$

.........(4.5)

$$\frac{d\rho}{\rho}$$ is the fractional loss of energy in the time $dt$.

Defining $dt =$ transit time of light between mirrors,

$L =$ mirrors separation

$$dt = L/c$$

from (4.5),

$$\tau_c = \frac{L}{c\alpha}$$

Inside the cavity, the net gain of intensity will be,

$$\left(\frac{dI}{dt}\right) = \frac{c_3\hbar\nu I (N_2 - N_1)}{8\pi\hbar\nu_3t_21}$$

And
\[
\left( \frac{dI}{dt} \right)_{\text{loss}} = \frac{I}{t_c}
\]

Leakage and undesired able absorption so, a necessary and sufficient condition for gain is,

\[
\left( \frac{dI}{dt} \right)_{\text{amp}} \geq \left( \frac{dI}{dt} \right)_{\text{loss}}
\]

\[
\left( \frac{dI}{dt} \right)_{\text{amp}} - \left( \frac{dI}{dt} \right)_{\text{loss}} \geq 0
\]

i.e we have

\[
(N_2 - N_1) \geq \frac{8\pi \nu^2 t_{21}}{C^3 t_c} \quad \text{.........(4.6)}
\]

This is called the threshold condition for sustained laser oscillation

At frequency \( \nu \).

Means, that we have to invert the population such that \( N_2 > N_1 \) by

\[
\frac{8\pi \nu^2 t_{21}}{C^3 t_{21}}
\]

Threshold power
Calculate how much power needed in the system to achieve the threshold condition for oscillation.

Need at least \((N_2 - N_1)\) excitations amount of energy = \((N_2 - N_1)\ h\nu\)

All these have to be achieved in a time which is less than \( t_{21} \) (spontaneous lifetime of atoms in the excited state)

Minimum power required,

\[
\rho_{\text{min}} = \frac{(N_2 - N_1) h\nu}{t_{21}}
\]
\[ \rho_{\text{min}} = \frac{8\pi}{c^3} \frac{v^2}{t_{c21}} \frac{h\nu}{t_{21}} \]

Finally

\[ \rho_{\text{min}} = \frac{8\pi h}{\lambda^3 t_c} \quad (4.7) \]

Note: power required \( \alpha \frac{1}{\lambda^3} \)

It is much more difficult to produce UV and x-ray laser than an IR laser.

Effect of degeneracy (multiplicity)
We have consider only the simplest case i.e atom with only a ground stated and an excited state.
In general, an atomic energy levels may consist of several distinct energy levels all having the same energy \( \Rightarrow \) degenerate of energy state

Describe by a number called the degeneracy of the state, \( g \).
\( g \Rightarrow \) is equal to the number of states superimposed in the level.

Instead of
\[ N_1 = e^{-\epsilon_1/kT} \quad \text{and} \]
\[ N_2 = e^{-\epsilon_2/kT} \quad \text{we have} \]

\[ \Rightarrow N_1 = g_1 A e^{-\epsilon_1/kT} \]
\[ N_2 = g_2 A e^{-\epsilon_2/kT} \]

So that, the threshold condition can be rewrite as,

\[ \left( N_2 - N_1 \frac{g_2}{g_1} \right) \geq \frac{8\pi}{c^3} \frac{v^2}{t_{c21}} \frac{h\nu}{t_{21}} \]
5.0 Spectral line width

5.1 Natural linewidth (Lorentzian line profile) (homogeneous broadening)

The lifetime $\Delta t_i$, the most probable time that the molecule is in the excited state, leads to an uncertainty $\Delta E$ in the energy of the state due to Hisenberg uncertainty principle, given by

$$\Delta E \cdot \Delta t_i \geq \frac{\hbar}{2\pi} \quad \text{…………..}(5.1)$$

This leads to the uncertainty of the emitted frequency,

$$\Delta \nu = \frac{\Delta E}{h} \geq \frac{1}{2\pi \Delta t_i} \quad \text{………..}(5.2)$$

This uncertainty in frequency causes the “natural” linewidth (as wide as the magnitude of the uncertainty)

$\Delta t_i$ is the lifetime of the excited state normally it is indicated as $\tau_i$ (or $\tau_{sp}$)

$$A = \frac{1}{\tau_i}$$

Equation (5.2) can be rewrite as

$$\Delta \nu \geq \frac{1}{2\pi \tau_i} \quad \text{………………..}(5.3)$$
Lorentzian lineshape

\[
\Delta \nu = \frac{\Delta \nu}{2\pi \left( (\nu_0 - \nu)^2 + \left( \frac{\Delta \nu}{2} \right)^2 \right)}
\]

With

\[
\Delta \nu = \frac{1}{2\pi \tau_i}
\]

Example

The natural linewidth of the sodium D\textsubscript{1} line at \( \lambda = 589.1 \)nm which corresponds to a transition from the \(^3\text{P}_{3/2}\) level (\( \tau_{sp} = 1 \)bns) to the \(^3\text{S}_{1/2}\) level ground state is

\[
\Delta \nu = \frac{1}{2\pi \times 1\text{bns}} = 10^7 \text{ s}^{-1} = 10 \text{ MHz}
\]
5.2 DOPPLER LINEWIDTH
(Inhomogeneous broading)

One of the major contributions to the spectral linewidth in gases at low pressure is the Doppler width.

Due to the thermal motion of the absorbing or emitting molecules.

Consider an excited molecule with a velocity, $\mathbf{v} = \{v_x, v_y, v_z\}$ relative to the rest frame of the observer.

Central freq of emission line is $\omega_o$ which is Doppler shifted to

$$\omega_e = \omega_o + \tilde{k} \cdot \tilde{D} \quad \text{.................(5.4)}$$

For an observer looking towards the emitting molecule.

- Apparent emission frequency $\omega_e$ is increased if the molecules moves towards the observer $(\tilde{k} \cdot \tilde{D} > 0)$
- Decreased if the molecule moves away $(\tilde{k} \cdot \tilde{D} < 0)$
ω, depend on the direction of \( \vec{k} \) and \( \vec{v} \)

Example

Molecule moves towards the observer, in the direction of the light propagation along +Z direction

\[ \vec{k} = \{0, 0, k_z\} \text{ and } (k) = \frac{2 \pi}{\lambda} \]

Giving rise to

\[ \omega_e = \omega_o + k_z \cdot v_z = \omega_o + \frac{2 \pi v_z}{\lambda} \] \hspace{1cm} (5.5)

From

\[ c = \nu \lambda \]
\[ \frac{c}{\lambda} = \nu \]
\[ \frac{1}{\lambda} = \frac{2 \pi \nu}{c} = \frac{\omega_o}{c} \]