Number of modes per unit volume of the cavity per unit frequency interval is given by:

\[ N_\nu = \frac{8\pi\nu^2}{c^3} \quad \cdots \quad (3.1) \]

Therefore, energy density (per unit freq. interval);

\[ U_\nu = \frac{8\pi\nu^3}{c^3} \frac{1}{e^{\frac{\nu}{kT}} - 1} \quad \cdots \quad (3.2) \quad \text{Unit of energy density: J/m}^3 \]

Plotting \( U_\nu \) against \( \nu \);

Eqn. 3.2 is important to derive Einstein Coefficients.

**Einstein Coefficients**

- Excited atom in free space has a lifetime of about \( 10^{-8} \) s.
- Equivalent to the average number of \( 10^8 \).
- Spontaneous transition from an excited state \( \rightarrow \) lower state per second.
- **Transition Probability** : The larger the transition rate, the greater the probability of transition.

Probability of spontaneous emission is called **Einstein A Coefficient**.

\[ A_{21} = \frac{1}{t_{21}} \quad \cdots \quad (3.3) \]

Definition: Is the probability per unit time per atom that the excited atom will decay spontaneously to the lower state.
For \( \text{N}_2 \) atom \( \rightarrow \text{N}_2 \text{A} \) transitions per unit time.

**Einstein B Coefficients**

Eg: \( B_{12} \) and \( B_{21} \)

\[ B_{12} = \text{is the absorption coefficient.} \]
\[ = \text{is the probability per atom per unit time per unit radiation density per unit frequency interval that absorption from (1) \( \rightarrow \) (2).} \]

Suppose that 2-level atoms are in an enclosure (a box), energy density (in the freq \( \nu \) to \( \nu + d\nu \)) = \( \rho(\nu)d\nu \). Then, probability per unit time that an atom will absorb a photon and be excited \( (1) \rightarrow (2) \).

So, the stimulated absorption per atom per unit time = \( B_{12} \rho(\nu) \)

for \( \text{N}_1 \) atoms = \( \text{N}_1 B_{12} \rho(\nu_{12}) \) \[ \text{[transition per unit time per unit freq interval]} \]

**Einstein \( B_{21} \) (for stimulated emission)**

\( B_{12} \rightarrow \text{The probability per atom per second per unit radiation energy density per unit frequency interval.} \)

For \( \text{N}_2 \) atoms, (per unit volume):

Transitions from \( (2) \rightarrow (1) = \text{N}_2 B_{21} \rho(\nu_{12}) \)
Relationship between A and B coefficients.

Consider the case where the system is in thermal equilibrium.

⇒ Total energy of the system must remain constant (no. of photons absorbed per second must be equal to the total no. emitted, stimulated and spontaneous)

Thus; \[ \frac{N_1 B_{12}}{\text{absorption}} \rho(v) = \frac{N_2 A_{21}}{\text{spontaneous}} + \frac{N_2 B_{21}}{\text{stimulated}} \rho(v) \] ........(3.4)

Solving for \( \rho(v) \),

\[ \rho(v) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} \] ........(3.5)

From Boltzmann distribution, at thermal equilibrium,

\[ \frac{N_1}{N_2} = e^{-\frac{h \nu}{kT}} \] ........(3.6)

Substitute (3.6) into (3.5),

\[ \rho(v) = \frac{A_{21}}{(e^\frac{h \nu}{kT})B_{12} - B_{21}} \] ........(3.7)

**Note:** This radiation density \( \rho(v) \) emitted at thermal equilibrium for 2-level system has to be identical to blackbody radiation density (Planck Law) given as;

\[ U_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h \nu}{kT}} - 1} \] ........(3.8)

Therefore, for them to be equal between (3.7) & (3.8), these relationship must hold,

\[ B_{12} = B_{21} \approx B \] ........(3.9)

\[ \frac{A_{21}}{B} = \frac{8\pi h \nu^3}{c^3} \] ........(3.10)
Note:

(3.9) shows that the probability of stimulated emission is equal to the probability of stimulated absorption.

(3.10) Ratio \( R \) of the rate of spontaneous emission to the rate of stimulated emission under thermal equilibrium.

\[ R = \frac{A_{21}}{\rho(\nu)B} \]

Using (3.7) into above,

\[ R = e^{\frac{hv}{kT}} - 1 \quad \ldots \ldots (3.11) \]

Substitute \( \nu = 5 \times 10^{14} \text{ Hz} \rightarrow \text{Green light.} \)

\[ R \approx 10^{-35} \left( \frac{\text{spon tan eous}}{\text{stimulated}} \right) \]

At thermal equilibrium, it is quite impossible to get the stimulated emission in visible region. But, for microwave…

\( \nu = 10^9 \text{ Hz} \quad R = 0.001 \)

Stimulated emission is dominant process, directionality in a microwave.

**For lasing action**, in visible region the population has to be inverted \( \rightarrow \) “population inversion” or negative temperature.

Why negative temperature?

\[ \frac{N_2}{N_1} = e^{-\frac{hv}{kT}} \quad \text{to make } N_2 > N_1, \text{ the term } \frac{hv}{kT} \text{ has to be positive, therefore} \]

this is possible when \( T \rightarrow (-T) \) so that; \( \frac{-hv}{k(-T)} \Rightarrow \frac{hv}{kT} \)
**Population inversion**; a situation whereby the system is no longer in thermal equilibrium and stimulated emission of visible light becomes possible.

**4.0 Threshold Condition.**

→ the **minimum** population difference $N_2 - N_1$ needed to sustain laser action.

A laser consists of an amplifying medium (that is being inverted) in the form of gas, liquid and solid placed between two mirrors.

Losses process in an optical resonator:

(i) Transmission, absorption and scattering by mirrors

(ii) Absorption within the amplifying medium due to other energy levels.

$$h\nu = E_5 - E_3$$
(iii) Scattering by optical inhomogeneities within the amplifying medium is important in solid-state lasers (impossible to produce perfect crystals)

(iv) Diffraction losses by the mirrors.

All this losses can be included in a parameter, $t_{\text{photon}}$.

$t_{\text{photon}}$: lifetime of a photon existing within the laser cavity.

**Deriving the threshold condition** - for lasing action.

2-level system:

\[
\begin{array}{c}
\rho(\nu) \\
\downarrow \\
\downarrow \\
E_1 \\
\hbar \nu \\
E_2
\end{array}
\]

Stimulated Emission rate per atom in state 2;

\[
W' = \rho(\nu)B = \frac{IB}{c} \quad \text{from;} \quad I(\nu) = \rho(\nu)c
\]

\((J/s)/cm^2 \quad J/cm^3.cm/s\)

From (3.10)

\[
W' = \frac{c^2I}{8\pi\hbar\nu^3}A \quad \text{ ..........(4.1)}
\]

$A$ = the probability per unit time of a spontaneous emission

\[A = \frac{1}{t_{21}} \quad (\text{related to the lifetime of the upper laser level})\]

\[
W' = \frac{c^2I}{8\pi\hbar\nu^3t_{21}} = \frac{\lambda^2I}{8\pi\hbar\nu t_{21}} = \frac{c^3\rho(\nu)}{8\pi\hbar\nu^3t_{21}} \quad \text{ ..........(4.2)}
\]

*W’ = transition rate per atom.*

\[\Rightarrow \text{For } N_2 \text{ atoms;} \quad W'N_2 = \frac{N_2c^2I}{8\pi\hbar\nu^3t_{21}} \quad \text{ ..........(4.3)}\]
Total absorption rate (for \( N_1 \) atoms in ground state);

\[
W'N_1 = \frac{N_1 c^2 I}{8\pi h \nu^3 t_{21}} \quad \text{(because } B_{12} = B_{21} = B \text{)}
\]

Consider the medium below;

Rate of change of intensity with time = \( \frac{dI}{dt} \)

= difference between stimulated emission and absorption rates.

i.e. = change of energy density per unit time \( xc \).

Or,

\[
\frac{dI}{dt} = h\nu[N_2 - N_1]\frac{c^2 I}{8\pi h \nu^3 t_{21}} - c \quad \text{.........(4.4)}
\]

For absorption,

\[
\begin{align*}
\left[ \frac{d\rho}{dt} \right] &= h\nu[N_2 - N_1]\frac{c^3 \rho(\nu)}{8\pi h \nu^3 t_{21}} \\
\frac{1}{c} \frac{dI}{dt} &= h\nu[N_2 - N_1]\frac{c^2 I/c}{8\pi h \nu^3 t_{21}}
\end{align*}
\]

If \( N_1 > N_2 \) (thermal equilibrium); \( \frac{dI}{dt} \) or \( \frac{dI}{dx} \) are negative.

⇒ As a result of \((N_2 - N_1)\) is negative.