Starting with spherically symmetric spacetime metric proposed by Morris-Thorne, the equations of wormhole structure are derived to solve Einstein Field equation using Morris-Thorne framework. The structure is obtained through embedding method where it can be shown that to maintain the curvature at the throat of wormhole, material with radial tension that exceed the energy density, \( r_0 > \rho c^2 \) is essential to expand and hold the throat from collapse. Matter that possesses this characteristic is coined as “exotic matter”, violates all the energy condition.

**Keywords:** Gravity, general relativity, Einstein equation, energy density.

**INTRODUCTION**

Prior to general relativity, special theory of relativity states that the space and time are combined to form single entities in 4-dimensional Minkowski spacetime which the geometry of spacetime is kept intact or static. General relativity then describes more dynamics geometric relationship of spacetime. The Einstein field equation represents this relationship and shows how the matter moves under the influence of curvature of spacetime and how the spacetime curve under the influence of matter \([1]\).

Wormhole arises as one of the solution in Einstein field equation. Begin in 1916, Karl Schwarzschild discovered the first exact solution for Einstein field equation, then in 1935 Einstein-Rosen Bridge or Schwarzschild wormhole which is the Schwarzschild geometry on hyperspace consist of bridge that connects the two asymptotically flat universes was mathematically discovered by Einstein together with Nathan Rosen \([2]\). Following this discovery, in 1988 K. S. Thorne together with his student Morris successfully “build” a new class of wormhole which possesses no event horizon, making this wormhole theoretically traversable not only for photon or subatomic particles but also for sizeable object as human \([3]\).

We consider the Morris-Thorne framework in constructing a traversable wormhole \([3]\) solution and derive the equation that describes the exoticity of matter used as the source to generate spacetime curvature. The following conditions listed by Morris and Thorne on characteristics of the traversable wormhole are followed:

1. Metric is spherically symmetric and static
2. In every point on the spacetime, the metric must fulfill Einstein field equation.
3. No event horizon.
4. The solution must be consist shape of wormhole; throat between two mouths that joined the asymptotically flat space and flare-out condition at minimum radius.
5. Matter or field that produces the curvature for wormhole must have physically acceptable energy.

One of our motivations to study the subject is because the solution for this type of wormhole is simple that they can be used as guide to understand basic general relativity, solve Einstein field equation, physically interpret the solution and explore the properties of the solution where we will also discuss briefly about the energy condition and phenomena that govern its violation.

**METHODOLOGY**

I. The Riemann, Ricci and Einstein Tensors

Starting from static spherically symmetric form of spacetime \((ct,r,\theta,\phi)\) coordinate is given as:

\[
ds^2 = -e^{2\Phi(r)}c^2dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)
\]

Where \(b(r)\) and \(\Phi(r)\) are arbitrary function of radial coordinate \(r\). Define \(b(r)\) as shape function that determine the shape of wormhole and \(\Phi(r)\) is the gravitational redshift. Radial coordinate \(r\) is nonmonotonic which covers the range \([r_0, +\infty)\) where \(r_0\) is minimum value at throat.

We write (1) in metric tensors \(g_{\mu\nu}\) form:

\[
g_{\mu\nu} = \text{diag}[g_{00}, g_{11}, g_{22}, g_{33}] \quad (2)
\]

And the conjugate metric tensor is \(g^{\mu\nu} = 1/g_{\mu\nu}\)

Christoffel symbols \(\Gamma^\alpha_{\beta\gamma}\) are given as:
\[ \Gamma^a_{\mu \nu} = g^{ab} \Gamma^b_{\mu \nu, \beta} = \frac{g^{ab}}{2} (\overline{\partial}_\mu g_{\nu \beta} + \overline{\partial}_\nu g_{\mu \beta} - \overline{\partial}_\beta g_{\mu \nu}) \]  

We have calculated the nonzero values for \( \Gamma^a_{\mu \nu} \):

\[
\begin{align*}
\Gamma^0_{01} &= \Phi' \\
\Gamma^1_{00} &= 2\Phi(1 - \frac{b}{r}) \\
\Gamma^1_{01} &= \frac{(b/r) - b'}{2(b - r)} \\
\Gamma^1_{22} &= b - r \\
\Gamma^1_{33} &= \cot \theta
\end{align*}
\]

while other components vanishes. The symbol prime denoting the differentiations with respect to radial coordinate \( r \). Riemann tensor \( R^a_{\mu \beta \nu} \):

\[ R^a_{\mu \beta \nu} = \overline{\partial}_\mu \Gamma^a_{\nu \beta} - \overline{\partial}_\nu \Gamma^a_{\mu \beta} + \Gamma^a_{\mu \lambda} \Gamma^\lambda_{\nu \beta} - \Gamma^a_{\nu \lambda} \Gamma^\lambda_{\mu \beta} \]  

Contraction of Riemann tensor yield Ricci tensor \( R_{\mu \nu} \):

\[ R_{\mu \nu} = \overline{\partial}_\mu \Gamma^a_{\nu \beta} - \overline{\partial}_\nu \Gamma^a_{\mu \beta} + \Gamma^a_{\mu \lambda} \Gamma^\lambda_{\nu \beta} - \Gamma^a_{\nu \lambda} \Gamma^\lambda_{\mu \beta} \]

We obtain the nonzero Ricci tensors by substitution of appropriate values in (4) into (6):

\[
\begin{align*}
R_{00} &= e^{2\Phi' - \frac{2b}{r}} \\
R_{11} &= \frac{(b/r - b')}{2(b - r)} \Phi' + \frac{2\Phi}{r} \\
R_{22} &= (b - r) \left( \Phi' + \frac{b'r - b}{2(r - b)} + b' \right) \\
R_{33} &= R_{22} \sin^2 \theta
\end{align*}
\]

The final contraction of Ricci tensor yield curvature scalar or Ricci scalar \( R = g^{\mu \nu} R_{\mu \nu} \):

\[
R = e^{2\Phi'} \left( \frac{b}{r} \Phi' + \frac{2\Phi}{r} + 4\Phi \Phi' \right) + \Phi' \frac{b - r}{r^2} + \Phi \frac{b'r - b}{r^2} + 2 \frac{b'}{r}
\]

Einstein curvature tensor \( G_{\mu \nu} \) relates Ricci tensor, Ricci scalar and metric tensor such as:

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \]  

Using the expression obtained from the Ricci curvature tensors and Ricci scalar, the nonzero Einstein tensors \( G_{\mu \nu} \) (12) are:

\[
\begin{align*}
G_{00} &= e^{2\Phi'} \frac{b'}{r} \\
G_{11} &= \frac{b - b' r}{r^2(b - r)} + \frac{2\Phi'}{r} + \frac{b'}{r(b - r)}
\end{align*}
\]

\[ G_{22} = \frac{b'}{2r} \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) \\
G_{33} = G_{33}(b - r) \]

\[ G_{12} = \frac{b}{2r} \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) \\
G_{23} = G_{23}(b - r) \]

II. Einstein Field Equation

Morris-Thorne [3] and Lemos et al.[4] simplified mathematical analysis and physical interpretation by introducing orthonormal basis vector that represents the proper reference frame for an observer who remains at rest in the coordinate system \((ct, r, \theta, \phi)\) with \((r, \theta, \phi)\) fixed [3,4]. Basis vectors in this coordinate system are denoted as \((\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3) = (\hat{e}_0, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)\) with transformation \( \tilde{e}_\mu = \Lambda^\nu_\mu e_\nu \) where

\[ \Lambda^\nu_\mu = \text{diag}[e^{\Phi}, (1 - b/r)^{1/2}, r^{-1}, (r \sin \theta)^{-1}] \]

Thus, we obtained orthonormal terms for metric tensors, Ricci tensors and Einstein tensors in basis vectors such as:

\[ g_{\mu \nu} = \Lambda^\mu_\nu \Lambda^\nu_\mu g_{\lambda \gamma} \]

\[ R_{\mu \nu} = \Lambda^\mu_\nu \Lambda^\nu_\mu R_{\lambda \gamma} \]

\[ G_{\mu \nu} = \Lambda^\mu_\nu \Lambda^\nu_\mu G_{\lambda \gamma} \]

The Einstein tensor in orthonormal reference frame is

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \]

The nonzero components:

\[ G_{00} = \frac{b'}{r^2} \\
G_{11} = \frac{2\Phi'}{r} \left( 1 - \frac{b}{r} \right) - \frac{b}{r^3} \\
G_{22} = \left( 1 - \frac{b}{r} \right) \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) - \frac{b}{r^3} \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) \]

Using the expression obtained from the Ricci curvature tensors and Ricci scalar, the nonzero Einstein tensors \( G_{\mu \nu} \) (12) are:

\[ G_{12} = \left( 1 - \frac{b}{r} \right) \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) - \frac{b}{r^3} \left( \frac{b'r - b}{r} \Phi' + \frac{2\Phi}{r^2} \right) \]

Birkhoff theorem testifies the existence of spherical (nontraversable) Schwarzschild wormhole as the only solution that can exists in vacuum Einstein field equation. Hence, the traversable wormhole must be intertwined by matter of field with nonzero stress-energy tensor. Einstein field equation demands the proportionality between the stress-energy tensor \( T_{\mu \nu} \) with Einstein curvature tensor \( G_{\mu \nu} \) giving us the
same algebraic structure between both tensors. The non-zero stress-energy tensors are $T_{00}, T_{11}, T_{22}$ and $T_{33}$ carry interpretation of measurement made by the static observer:

$$T_{ii} = \rho(r)c^2$$  
(26)

$$T_{ii} = -\tau(r)$$  
(27)

$$T_{ii} = T_{ii} = p(r)$$  
(28)

in which “$\rho(r)$ is the total energy density, $\tau(r)$ is the radial tension and $p(r)$ is the pressure measured in directions of orthogonal to radial direction” [3,4].

Using relationship (17) with (26-28) to obtain:

$$\rho(r) = \frac{c^2}{8\pi G} \frac{b'}{r^2}$$  
(29)

$$\tau(r) = \frac{c^2}{8\pi G} \left[ \frac{b}{r} - 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right]$$  
(30)

$$p(r) = \frac{c^2}{8\pi G} \left( 1 - \frac{b}{r} \right) \frac{\Phi'' - \frac{b'r-b}{2r^2(1-b/r)} \Phi'}{r}$$  
(31)

Differentiate (30) with respect to radial coordinate $r$ then eliminate $b'$ and $\Phi''$ in (29) and (31) to obtain relationship:

$$\tau = \left( \frac{\rho - \tau}{r} \right)$$  
(32)

which is “Euler equation or equation of hydrostatic equilibrium for material threading the wormhole”[4]. From (24), rewrite (23) as:

$$\rho(r) = \frac{r}{2} \left[ \rho - \tau \right]$$  
(33)

Thus we obtain three differential equations that correlate with five unknown functions of $\rho, \rho, \tau, \Phi$ and $b$. From (29-31) function $\rho$ is determined by $b(r)$ while $\tau$ is determined by $b(r)$ and $\Phi'(r)$. Normal way of solving this type of Einstein field equations is by assuming specific matter or field that contributes to the source of stress-energy tensor. From the nature of this source, we derived the equation of state of radial tension and tangential pressure as a function of energy density. Then we obtained two equations of state and three field equations that yield five equations with five unknown functions that is sufficient to generate geometry of spacetime given in metric $g_{\mu\nu}$. By following Morris and Thorne framework, we first generate suitable geometry for wormhole solution by “controlling” shape function $b$ and redshift function $\Phi$. Then we determine the favorable stress-energy tensor from the geometry through the field equations of $\rho, p$ and $\tau$ [5].

### III. Mathematical of Embedding

We represent the shape of wormhole via the embedding diagram and obtain the information of our chosen shape function through it. An equatorial slice, $\theta = \pi/2$ through spherical symmetry geometry at the fixed $t = const$ is considered. Then the line element represents this slice is re-parameterized by using (1):

$$ds^2 = \frac{dr^2}{1-b/r} + r^2 d\phi^2$$  
(34)

Our strategies are to construct a mental image of structure for this slice by embedding this metric into 3-D Euclidean space. We write metric in this surface in cylindrical coordinate $(r, \phi, z)$ as:

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$  
(35)

Due to the symmetry around the axis, equation for the embedded surface is assigned by single equation $z = z(r)$, which results to the metric of this surface:

$$ds^2 = \left[ 1 + \left( \frac{dz}{dr} \right)^2 \right] dr^2 + r^2 d\phi^2$$  
(36)

Equating (34) with (36), we obtained equation of embedding surface:

$$\frac{dz}{dr} = \pm \left( \frac{r}{b(r)} - 1 \right)^{-1/2}$$  
(37)

![FIGURE 1. Embedding 2-dimensional curve profile of wormhole near throat region in 3-dimensional Euclidean space. Wormhole shape is generated by rotating curve about 2π through z-axis](image)

Since radial coordinate $r$ behave poorly near the throat, we introduce the proper radial distance $l$ to describe the divergence of throat i.e.: upper part of wormhole $z > 0$

$$l(r) = \frac{a}{\left( 1-b(r)/r \right)^{1/2}}$$  
(38)

and lower part of wormhole $z < 0$

$$l(r) = -\frac{a}{\left( 1-b(r)/r \right)^{1/2}}$$  
(39)
Lower limit of the integration is the radius of throat, \( r_0 \) and the maximum upper limit is set up as the radius of wormhole mouth, \( a \) as presented in Fig 2. Shape function \( b(r) \) is positive so that \( \sqrt{r/b-1} \) is real. Solution for wormhole required the ‘flaring-out’ condition at the throat. From (37) we obtain inverse embedding function \( r'(z) \) as:

\[
\frac{dr}{dz} = \pm \left( \frac{r}{b(r)} - 1 \right)^{1/2}
\]  

(40)

The geometry of flare-out condition is applied by strict minimalit at the wormhole throat which is:

\[
\frac{d^2r}{dz^2} = \frac{b-b'r}{2b^2} > 0
\]  

(41)

![Image](image_url)

**FIGURE 2.** Embedding diagram for geometry of wormhole that separating two parts of universe.

IV. Absence of Horizon

Setting redshift, \( \Phi(r) \) everywhere finite as \( l \to \pm \infty \) to eliminate event horizon means that the proper time lapse vanished at the finite coordinate time.

V. Exotic Matter

At the throat of a wormhole, \( r=b=r_0 \), the energy density \( \rho \) is finite with \( b' \) (29) and \( (1-b/r)\Phi' \to 0 \). From (30) we obtain tension in wormhole throat. Further calculation (3) reveals the huge 9 magnitude of tension approximately equal to pressure inside the most massive neutron star for \( b_0 \approx 3km \)

As the result, the dimensionless function is introduced to define the configuration required to support the wormhole i.e. \( \xi = (\tau - \rho \chi^2) / \rho \chi^2 \). From the substitution (29, 30, and 33) into this dimensionless function, we obtain:

\[
\xi = \frac{\tau - \rho \chi^2}{\rho \chi^2} = \left( \frac{b'/r - 2(r-b)\Phi' - b'}{b'} \right)
\]  

(42)

Comparing (44) with (43):

\[
\xi = \frac{2b^2}{|b|} \left( \frac{d^2r}{|b| dz^2} \right) - 2r \left( \frac{1-b}{r} \right) \Phi'
\]  

(43)

Again at the throat of wormhole, \( \rho \) is finite, lead to the finiteness of \( b' \) and \( (1-b/r)\Phi' \to 0 \) give us relationship of:

\[
\xi_0 = \frac{\tau_0 - \rho_0 \chi^2}{\rho_0 \chi^2} > 0
\]  

(44)

which is defined as the exoticity function of material.

**RESULTS AND ANALYSIS**

The constraint in \( \tau_0 > \rho_0 \chi^2 \) is very clamorous since it states that the radial tension at throat is large as it must surpassed the magnitude of the energy density. The material that exhibits property of \( \tau > \rho \chi^2 > 0 \) is called “exotic matter”. This property means the measurement is made by the observer who moves through throat with radial velocity approaching speed of light, \( \gamma \geq 1 \). Let the basis vectors for this static observer \( \vec{e}_{0i}, \vec{e}_{1i}, \vec{e}_{2i}, \vec{e}_{3i} \) applied to the standard Lorentz transformation. We are interested with the observation of energy density measured by the observer’s time. It is represented by the projection of stress-energy tensor (31) with the basis vector on the observer’s time obtained by \( \vec{e}_{0i} = \vec{e}_{00} \pm \gamma(v/c)\vec{e}_{1i} \):

\[
T_{00} = \gamma^2 T_{10} + (v/c)^2 T_{0i} + \gamma^2 (v/c)^2 T_{ii}
\]

\[
= \gamma^2 \left( \rho_0 \chi^2 - (v/c)^2 \tau_0 \right)
\]

\[
= \gamma^2 \left( \rho_0 \chi^2 - \tau_0 \right) + \tau_0
\]  

(45)

where \( \gamma = (1-v^2/c^2)^{-1/2} \). As the observer move with very fast radial velocity (\( \gamma \) very large) he will observed negative energy density. This is due to null geodesic that enter wormhole at “upper” mouth and emerge at “lower” having nonmonotonic cross section area. This situation only happens if the matter in this region exhibit gravitational repulsion properties, which repulse the passing light ray. A repulsion required negative energy density which only possible if the null energy condition, NEC [3, 4, 6] is violated. The negative mass energy acts like divergent lens that display negative angle of deflection [7]. Amount of exotic matter used in wormhole solution is quantified via the \( \xi(r) = \left( \tau - \rho \chi^2 \right)^2 \) Due to the problematic nature of the exotic matter we must minimize our usage of this matter in wormhole solution.

**DISCUSSION**

VI. Energy Condition

The clamorous point of exotic matter leads us to minimize usage of this matter in wormhole solution
by applying some constraint on energy density. The energy condition can be generalized into three types; namely weak energy condition (WEC), strong energy condition (SEC) and null energy condition (NEC). NEC is the weakest energy condition that can be satisfied by any classical matter. Many mathematical theorems in general relativity are built based on the assumption that classical matter must satisfy all energy condition mentioned. However, we also can show the violation of this energy condition [10] by considering the average weak energy condition (AWEC) that is taking the average point wise energy condition along the observer’s geodesic to obtain appropriate solution for Einstein field equation. Instead of measuring the energy density at one point in observer’s geodesics, we averaged all the energy density measured across the geodesics. This allows the existence of negative energy at some portion of the geodesics as long as the average sum of this energy density is positive. We also can use the same definition for average null energy condition (ANECE) and average strong energy condition (ASEC).

By quantum inequalities (QI), the inertial observer at Minkowski spacetime will observe that the amount and duration of negative energy density is limited [9]; this observer can only see the large negative energy density during a short period of time which is the suitable condition in maintaining wormhole shape. However, QI is invalid for the curve spacetime. If we consider small spacetime volume at the throat of wormhole, then the spacetime can be approximated as flat surface, so that QI can be applied. From [11] analysis, the throat possesses by wormhole is slightly larger than Planck size.

VII. Violation

The violation of the energy condition has been observed in both classical and quantum level [10] by phenomenon such as Casimir effect, Hawking evaporation, “squeezed vacuum state” in non-linear optics and many others. Cosmologically it has been observed [12] that the cosmological fluid violates SEC which lead to the possibility that NEC is also violated in this regime. These results play a vital role in the construction of wormholes since the quantum effect has influences on the stress-energy tensor and semiclassical gravity of wormhole solution satisfied QI. Hence the at the quantum regime, these exotic spacetime arise naturally.

CONCLUSION

In conclusion, by considering a spherical symmetry spacetime metric we obtained important equation of spacetime structure in solving Einstein field equation. Following Morris and Thorne approach in solving these equations yield the definition of dimensionless function $\xi$ known as exoticity function

$$\xi_0 = \frac{\mathcal{F}_0 - \rho c^2}{\rho c^2} > 0$$

This implies troublesome constraints on properties of material at the throat of wormhole where it must exhibit negative energy density. The exoticity of this material represents the measure of energy conditions violation.

ACKNOWLEDGEMENT

We thank Mr. Anuar Alias and Dr Bijan Nikouravan for the productive comments, discussion, criticism and idea.

REFERENCES